Stability of Heterogeneous Aeolotropic Cylindrical Shells under Combined Loading

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A theoretical analysis of the buckling problems of heterogeneous aeolotropic cylindrical shells under combined axial, radial, and torsional loads is presented. Four boundary conditions at each end of the cylinder are satisfied for the case of both ends hinged or that of both ends clamped. Classical thin shell theory of small deflection is followed. Because only six elastic coefficients are required out of the usual 21 for a general aeolotropic body, it is possible to solve Flugge's differential equations of equilibrium by assuming suitable functions for the displacements of the middle surface. By the superposition of these solutions, a general solution that satisfies the boundary conditions can be reached. If the thin shell is laminated from layers of different materials, the resultant forces and moments of an element are integrated from layer to layer by considering that the six elastic coefficients are piecewise continuous. Orthotropic and isotropic materials are particular cases of this analysis.

Nomenclature

a	= radius to middle surface of shell
$A, \bar{A}, B, \bar{B}, C, D, \bar{D}$	= elastic coefficients
$e, \bar{e}$	= strain components
e^0	= strain component of middle surface
E	= Young's modulus
G	= shear modulus
h	= thickness of cylindrical shell
k	= buckling coefficient
L	= length of cylinder
m	= number of half-waves in axial direction
M	= resultant moment per unit length
n	= number of waves in circumferential direction
N	= resultant force per unit length
p	= external radial pressure
P	= external axial compression per unit length
q	= load parameter
T	= external torsional force per unit length
u	= axial displacement of middle surface
v	= circumferential displacement of middle surface
w	= radial displacement of middle surface
x	= axial coordinate of middle surface
X	= change of curvature or angle of twist of middle surface
z	= radial coordinate of middle surface
θ	= circumferential coordinate of middle surface
λ	= dimensionless parameter = $m\pi a/L$
ν	= Poisson's ratio
$\sigma, \bar{\sigma}$	= stress components
ξ, η	= major and minor elastic axes of orthotropic material parallel to x-θ plane
ϕ	= angle between x axis and ξ axis
$(\ )_{,\alpha}$	= $\partial(\ )/\partial\alpha$, $\alpha = x, \theta, z$

Introduction

IT seems that so far there is no publication that covers a general study of the stability of aeolotropic (or anisotropic) material under combined loadings. Ambartsumian,[1] in his recent survey, cited several papers, most of which are in Russian. However, these papers actually deal with orthotropic material. March et al.[12] solved the buckling of thin-walled plywood cylinders in torsion in 1945, and March[13] solved the buckling of long thin plywood cylinders in axial compression in 1956. The former is based on the thin shell theory of small deflection, and the latter is based on the thin shell theory of large deflection. Both use the energy method. They actually deal with the heterogeneous (or nonhomogeneous) and anisotropic material, because each layer of the plywood can be orientated at any angle. However, these analyses still are limited to single loading and partially satisfied boundary conditions. As for the papers by Becker and Gerard[2] in 1962, by Cheng[3] in 1961, by Hess[9] in 1961, and by Thichemann et al.[4] in 1960, they also are limited to orthotropic material and partially satisfied boundary conditions.

The present analysis is more general in solving the buckling problems of nonhomogeneous anisotropic cylindrical shells under combined axial, radial, and torsional loads with all four boundary conditions at each end of the cylinder satisfied. Based on the classical thin shell theory of small deflection, the usual assumptions[14] can be summarized as follows:

1) The ratio of the thickness of the shell to the radii of curvature of its middle surface is small as compared with unity.

2) Displacements are very small as compared with the thickness of the shell.

3) The straight fibers of an element which are perpendicular to the middle surface before deformation remain so after deformation and do not change their length.

Because of the third assumption, it is possible to reduce the number of independent elastic coefficients C_{ij} of an anisotropic thin shell from 21 to 6. With the elastic coefficients given, stresses can be defined as functions of the displacements of the middle surface from the stress-strain relations and strain-displacement relations. Furthermore, the resultant forces and moments can be defined as functions of the displacements of the middle surface. If the shell is laminated from layers of different materials, it is then piecewise homogeneous but nonhomogeneous as a whole. Similarly, C_{ij}'s are piecewise continuous. The resultant forces and moments can be integrated piece by piece.

Flugge's differential equations of equilibrium[6] are used to derive the characteristic equation for buckling under combined loads. Donnell's method[5] is followed to establish the boundary conditions. Since there are so many coefficients involved, the electronic computer should be used to solve the problem. Because of this fact, the authors do not at-

Received by IAS August 13, 1962; revision received January 24, 1963.
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tempt to simplify anything in the course of derivation of equations.

As has been pointed out already by Von Kármán and Tsien,[15] Gerard and Becker,[8] and others, large deflection theory must be used for buckling under axial compression. Hence, in the application of this analysis, axial load should be limited to tension or small compression in combination with other loads, e.g., torsion and internal pressure.

When materials like plywood, reinforced structure, or glass-reinforced plastic are layered orthotropically, they are nonhomogeneous and orthotropic. If they are not layered orthotropically, they are nonhomogeneous and anisotropic. In this paper, when cylindrical shells laminated from orthotropic materials, the inclination angle ϕ_i between the major elastic axis of any ith layer and the axial axis of the cylinder can be any value. The thickness of each layer does not necessarily have to be equal to that of the other layers.

Basic Relations and Equations

For infinitesimal strains, the stress-strain relations may be expressed by the general Hooke's law[7, 11] in the matrix form:

$$\begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_z \\ \sigma_{\theta z} \\ \sigma_{xz} \\ \sigma_{x\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_\theta \\ e_z \\ e_{\theta z} \\ e_{xz} \\ e_{x\theta} \end{bmatrix} \tag{1}$$

where the C_{ij}'s ($i, j = 1, 2, \ldots, 6$) are elastic coefficients, or moduli, of the material. Also, since $C_{ij} = C_{ji}$, there are 21 elastic coefficients for an aeolotropic or anisotropic material. Based on assumption 3 mentioned in the introduction, the stress-strain relations of an anisotropic material become

$$\begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{bmatrix} e_x \\ e_\theta \\ e_{x\theta} \end{bmatrix} \tag{2}$$

There are now only six elastic coefficients in Eqs. (2). With these coefficients given, the resultant force (moment) and stress relations[6] can be established from Fig. 1:

$$\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x \left(1 + \frac{z}{a}\right) dz \\ N_\theta &= \int_{-h/2}^{h/2} \sigma_\theta dz \\ N_{x\theta} &= \int_{-h/2}^{h/2} \sigma_{x\theta} \left(1 + \frac{z}{a}\right) dz \\ N_{\theta x} &= \int_{-h/2}^{h/2} \sigma_{\theta x} dz \\ M_x &= \int_{-h/2}^{h/2} \sigma_x \left(1 + \frac{z}{a}\right) z dz \\ M_\theta &= \int_{-h/2}^{h/2} \sigma_\theta z dz \\ M_{x\theta} &= \int_{-h/2}^{h/2} \sigma_{x\theta} \left(1 + \frac{z}{a}\right) z dz \\ M_{\theta x} &= \int_{-h/2}^{h/2} \sigma_{\theta x} z dz \end{aligned} \tag{3}$$

For a cylindrical element, the strains e_x, e_θ, and $e_{x\theta}$ can be expressed in terms of the displacements u, v, and w of the middle surface of the shell:[6]

$$\begin{aligned} e_x &= u_{,x} - z w_{,xx} \\ e_\theta &= \frac{v_{,\theta}}{a} - \frac{z}{a} \frac{w_{,\theta\theta}}{a + z} + \frac{w}{a + z} \\ e_{x\theta} &= \frac{u_{,\theta}}{a + z} + \left(1 + \frac{z}{a}\right) v_{,x} - w_{,x\theta} \left(\frac{z}{a} + \frac{z}{a + z}\right) \end{aligned} \tag{4}$$

If $1/(a + z)$ is expanded into the power series

$$\frac{1}{a + z} = \frac{1}{a} \left(1 - \frac{z}{a} + \left(\frac{z}{a}\right)^2 - \left(\frac{z}{a}\right)^3 + \ldots\right) \tag{5}$$

and $(z/a)^3$ or higher orders are neglected as compared with 1, Eqs. (4) become

$$\begin{aligned} e_x &= e_x^0 + z X_x \\ e_\theta &= e_\theta^0 + z[1 - (z/a)] X_\theta \\ e_{x\theta} &= [1 + (z^2/2a^2)]\, e_{x\theta}^0 + z[1 - (z/2a)] X_{x\theta} \end{aligned} \tag{6}$$

where

$$\begin{aligned} e_x^0 &= u_{,x} \\ e_\theta^0 &= (1/a)(v_{,\theta} + w) \\ e_{x\theta}^0 &= (u_{,\theta}/a) + v_{,x} \end{aligned} \tag{7}$$

are the strains at the middle surface of the shell, and

$$\begin{aligned} X_x &= -w_{,xx} \\ X_\theta &= -(1/a^2)(w_{,\theta\theta} + w) \\ X_{x\theta} &= -(2/a)[w_{,x\theta} + (u_{,\theta}/2a) - (v_{,x}/2)] \end{aligned} \tag{8}$$

are the parameters corresponding to the change of curvatures or the change of angle of twist.

Substituting Eqs. (2) and (6) in Eqs. (3), the following resultant force (moment) and displacement relations are established:

$$\begin{bmatrix} N_x \\ N_\theta \\ N_{x\theta} \\ N_{\theta x} \\ M_x \\ M_\theta \\ M_{x\theta} \\ M_{\theta x} \end{bmatrix} = \begin{bmatrix} A_{11} + \frac{B_{11}}{a} & A_{12} + \frac{B_{12}}{a} & A_{16} + \frac{B_{16}}{a} + \frac{D_{16}}{2a^2} & B_{11} + \frac{D_{11}}{a} & B_{12} & B_{16} + \frac{D_{16}}{2a} \\ A_{12} & A_{22} & A_{26} + \frac{D_{26}}{2a^2} & B_{12} & B_{22} - \frac{D_{22}}{a} & B_{26} - \frac{D_{26}}{2a} \\ A_{16} + \frac{B_{16}}{a} & A_{26} + \frac{B_{26}}{a} & A_{66} + \frac{B_{66}}{a} + \frac{D_{66}}{2a^2} & B_{16} + \frac{D_{16}}{a} & B_{26} & B_{66} + \frac{D_{66}}{2a} \\ A_{16} & A_{26} & A_{66} + \frac{D_{66}}{2a^2} & B_{16} & B_{26} - \frac{D_{26}}{a} & B_{66} - \frac{D_{66}}{2a} \\ B_{11} + \frac{D_{11}}{a} & B_{12} + \frac{D_{12}}{a} & B_{16} + \frac{D_{16}}{a} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} + \frac{D_{16}}{a} & B_{26} + \frac{D_{26}}{a} & B_{66} + \frac{D_{66}}{a} & D_{16} & D_{26} & D_{66} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} e_x^0 \\ e_\theta^0 \\ e_{x\theta}^0 \\ X_x \\ X_\theta \\ X_{x\theta} \end{bmatrix} \tag{9}$$

where, respectively,

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C_{ij}(1, z, z^2)dz \qquad (10)$$

with $i,j = 1, 2, 6$. A_{ij}'s, B_{ij}'s, and D_{ij}'s are symmetric coefficients because C_{ij}'s are symmetric. If the thin shell is made from laminated layers, then the integration of Eqs. (10) can be carried out from layer to layer with C_{ij} to be constant for each layer; C_{ij} may be different from layer to layer.

From Fig. 1, considering the deformation of an infinitesimal element, the following differential equations of equilibrium exist during buckling:[6]

$$aN_{x,x} + N_{\theta x,\theta} - p(u_{,\theta\theta} - aw_{,x}) - aPu_{,xx} - 2Tu_{,x\theta} = 0$$

$$aN_{\theta,\theta} + a^2N_{x\theta,x} + M_{\theta,\theta} + aM_{x\theta,x} - pa(v_{,\theta\theta} + w_{,\theta}) - a^2Pv_{,xx} - 2aT(v_{,x\theta} + w_{,x}) = 0 \qquad (11)$$

$$M_{\theta,\theta\theta} + a(M_{x\theta} + M_{\theta x})_{,x\theta} + a^2M_{x,xx} - aN_\theta - pa(au_{,x} - v_{,\theta} + w_{,\theta\theta}) - a^2Pw_{,xx} + 2aT(v_{,x} - w_{,x\theta}) = 0$$

where p, P, and T are, respectively, the external radial pressure, the external axial compression per unit length, and the external torsional (shearing) force per unit length. Substituting Eqs. (7–9) in Eqs. (11), the differential equations of equilibrium then can be expressed in terms of the displacements of the middle surface of the shell:

$$a^2u_{,xx}(\bar{A}_{11} + \bar{B}_{11} - q_2) + 2au_{,x\theta}(\bar{A}_{16} - q_3) + u_{,\theta\theta}(\bar{A}_{66} - \bar{B}_{66} + \bar{D}_{66} - q_1) + a^2v_{,xx}(\bar{A}_{16} + 2\bar{B}_{16} + \bar{D}_{16}) + av_{,x\theta}(\bar{A}_{12} + \bar{A}_{66} + \bar{B}_{12} + \bar{B}_{66}) + v_{,\theta\theta}\bar{A}_{26} - a^3w_{,xxx}(\bar{B}_{11} + \bar{D}_{11}) - a^2w_{,xx\theta}(3\bar{B}_{16} + \bar{D}_{16}) - aw_{,x\theta\theta}(\bar{B}_{12} + 2\bar{B}_{66} - \bar{D}_{66}) - w_{,\theta\theta\theta}(\bar{B}_{26} - \bar{D}_{26}) + aw_{,x}(\bar{A}_{12} + q_1) + w_{,\theta}(\bar{A}_{26} - \bar{B}_{26} + \bar{D}_{26}) = 0 \qquad (12a)$$

$$a^2u_{,xx}(\bar{A}_{16} + 2\bar{B}_{16} + \bar{D}_{16}) + au_{,x\theta}(\bar{A}_{12} + \bar{A}_{66} + \bar{B}_{66} + \bar{B}_{12}) + u_{,\theta\theta}\bar{A}_{26} + a^2v_{,xx}(\bar{A}_{66} + 3\bar{B}_{66} + 3\bar{D}_{66} - q_2) + 2av_{,x\theta}(\bar{A}_{26} + 2\bar{B}_{26} + \bar{D}_{26} - q_3) + v_{,\theta\theta}(1 + \bar{B}_{22} - q_1) - a^3w_{,xxx}(\bar{B}_{16} + 2\bar{D}_{16}) - a^2w_{,xx\theta}(\bar{B}_{12} + 2\bar{B}_{66} + \bar{D}_{12} + 3\bar{D}_{66}) - aw_{,x\theta\theta}(3\bar{B}_{26} + 2\bar{D}_{26}) - w_{,\theta\theta\theta}\bar{B}_{22} + aw_{,x}(\bar{A}_{26} + \bar{B}_{26} - 2q_3) + w_{,\theta}(1 - q_1) = 0 \qquad (12b)$$

$$-a^3u_{,xxx}(\bar{B}_{11} + \bar{D}_{11}) - a^2u_{,xx\theta}(3\bar{B}_{16} + \bar{D}_{16}) - au_{,x\theta\theta}(\bar{B}_{12} + 2\bar{B}_{66} - \bar{D}_{66}) - u_{,\theta\theta\theta}(\bar{B}_{26} - \bar{D}_{26}) + au_{,x}(\bar{A}_{12} + q_1) + u_{,\theta}(\bar{A}_{26} - \bar{B}_{26} + \bar{D}_{26}) - a^3v_{,xxx} \times (\bar{B}_{16} + 2\bar{D}_{16}) - a^2v_{,xx\theta}(\bar{B}_{12} + 2\bar{B}_{66} + \bar{D}_{12} + 3\bar{D}_{66}) - av_{,x\theta\theta}(3\bar{B}_{26} + 2\bar{D}_{26}) - v_{,\theta\theta\theta}\bar{B}_{22} + av_{,x}(\bar{A}_{26} + \bar{B}_{26} - 2q_3) + v_{,\theta}(1 - q_1) + a^4w_{,xxxx}\bar{D}_{11} + 4a^3w_{,xxx\theta}\bar{D}_{16} + 2a^2w_{,xx\theta\theta}(\bar{D}_{12} + 2\bar{D}_{66}) + 4aw_{,x\theta\theta\theta}\bar{D}_{26} + w_{,\theta\theta\theta\theta}\bar{D}_{22} - a^2w_{,xx}(2\bar{B}_{12} - q_2) - 2aw_{,x\theta}(2\bar{B}_{26} - \bar{D}_{26} - q_3) - w_{,\theta\theta}(2\bar{B}_{22} - 2\bar{D}_{22} - q_1) + w(1 - \bar{B}_{22} + \bar{D}_{22}) = 0 \qquad (12c)$$

where, respectively,

$$(\bar{A}_{ij}, \bar{B}_{ij}, \bar{D}_{ij}) = (1/A_{22})(A_{ij}, B_{ij}/a, D_{ij}/a^2) \qquad (13)$$

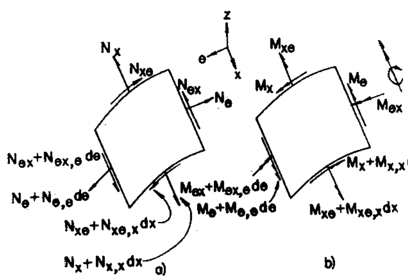

Fig. 1 Shell element

and

$$(q_1, q_2, q_3) = (1/A_{22})(pa, P, T) \qquad (14)$$

with $i,j = 1, 2, 6$.

Solution of Differential Equations

Equations (12) are complicated partial differential equations that are very difficult to solve exactly. One possible particular solution is by the inverse method. Let the displacements of the middle surface of the shell be

$$\begin{aligned} u &= U \sin[(\lambda x/a) + n\theta] \\ v &= V \sin[(\lambda x/a) + n\theta] \\ w &= W \cos[(\lambda x/a) + n\theta] \end{aligned} \qquad (15)$$

where U, V, and W are constants, $\lambda = m\pi a/L$, n is the number of waves in the circumferential direction, and m is the number of half-waves in the axial direction if the circumferential waves do not spin along the cylinder.

Equations (15), of which the origin of the coordinates is at the mid-length of the middle surface of the cylindrical shell as shown in Fig. 2, cannot satisfy any boundary condition. If the length L of the cylinder is very long, then the constraints at the ends will not affect greatly the magnitude of the critical stresses. If the cylinder is not very long, then the end conditions must be considered.

First, let the solution of Eqs. (15) be investigated. Substituting Eqs. (15) in Eqs. (12), one has

$$\begin{bmatrix} F_{11} - \lambda^2q_2 - 2n\lambda q_3 - n^2q_1 & F_{12} & F_{13} + \lambda q_1 \\ F_{12} & F_{22} - \lambda^2q_2 - 2n\lambda q_3 - n^2q_1 & F_{23} - 2\lambda q_3 - nq_1 \\ F_{13} + \lambda q_1 & F_{23} - 2\lambda q_3 - nq_1 & F_{33} - \lambda^2q_2 - 2n\lambda q_3 - n^2q_1 \end{bmatrix} \times \begin{bmatrix} U \\ V \\ W \end{bmatrix} = 0 \qquad (16)$$

where

$$\begin{aligned}
F_{11} &= (\bar{A}_{11} + \bar{B}_{11})\lambda^2 + 2n\bar{A}_{16}\lambda + n^2(\bar{A}_{66} - \bar{B}_{66} + \bar{D}_{66}) \\
F_{12} &= (\bar{A}_{16} + 2\bar{B}_{16} + \bar{D}_{16})\lambda^2 + n(\bar{A}_{12} + \bar{A}_{66} + \bar{B}_{12} + \bar{B}_{66})\lambda + n^2\bar{A}_{26} \\
F_{13} &= (\bar{B}_{11} + \bar{D}_{11})\lambda^3 + n(3\bar{B}_{16} + \bar{D}_{16})\lambda^2 + [n^2(\bar{B}_{12} + 2\bar{B}_{66} - \bar{D}_{66}) + \bar{A}_{12}]\lambda + n^3(\bar{B}_{26} - \bar{D}_{26}) + n(\bar{A}_{26} - \bar{B}_{26} + \bar{D}_{26}) \\
F_{22} &= (\bar{A}_{66} + 3\bar{B}_{66} + 3\bar{D}_{66})\lambda^2 + 2n(\bar{A}_{26} + 2\bar{B}_{26} + \bar{D}_{26})\lambda + n^2(1 + \bar{B}_{22}) \\
F_{23} &= (\bar{B}_{16} + 2\bar{D}_{16})\lambda^3 + n(\bar{B}_{12} + 2\bar{B}_{66} + \bar{D}_{12} + 3\bar{D}_{66})\lambda^2 + [n^2(3\bar{B}_{26} + 2\bar{D}_{26}) + \bar{A}_{26} + \bar{B}_{26}]\lambda + n^3\bar{B}_{22} + n \\
F_{33} &= \bar{D}_{11}\lambda^4 + 4n\bar{D}_{16}\lambda^3 + 2[n^2(\bar{D}_{12} + 2\bar{D}_{66}) + \bar{B}_{12}]\lambda^2 + 2n(2n^2\bar{D}_{26} + 2\bar{B}_{26} - \bar{D}_{26})\lambda + (n^2 - 1)^2\bar{D}_{22} + (2n^2 - 1)\bar{B}_{22} + 1
\end{aligned} \qquad (17)$$

In order to have nontrivial solution of Eqs. (16), the determinant of the coefficient matrix must be equal to zero:

$$\begin{vmatrix} F_{11} - \lambda^2q_2 - 2n\lambda q_3 - n^2q_1 & F_{12} & F_{13} + \lambda q_1 \\ F_{12} & F_{22} - \lambda^2q_2 - 2n\lambda q_3 - n^2q_1 & F_{23} - 2\lambda q_3 - nq_1 \\ F_{13} + \lambda q_1 & F_{23} - 2\lambda q_3 - nq_1 & F_{33} - \lambda^2q_2 - 2n\lambda q_3 - n^2q_1 \end{vmatrix} = 0 \qquad (18)$$

Equation (18) can be expanded into the following quadratic form by neglecting those terms containing the factors $q_iq_jq_k$, where $i, j, k = 1, 2, 3$, which are negligibly small as compared with the constant term H_6:

$$H_1q_1^2 + H_2q_1q_3 + H_3q_3^2 + H_4q_1 + H_5q_3 + H_6 + H_7q_2^2 + H_8q_1q_2 + H_9q_2q_3 + H_{10}q_2 = 0 \qquad (19)$$

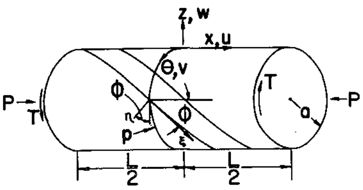

Fig. 2 Orientation of axes in the middle surface of cylindrical shell

where

$$
\begin{aligned}
H_1 &= n^4(F_{11} + F_{22} + F_{33}) - 2n\lambda F_{12} + 2n^2\lambda F_{13} - \\
&\quad \lambda^2 F_{22} - 2n^3 F_{23} - n^2 F_{11} \\
H_2 &= 4\lambda[n^3(F_{11} + F_{22} + F_{33}) - \lambda F_{12} + n\lambda F_{13} - \\
&\quad nF_{11} - 2n^2 F_{23}] \\
H_3 &= 4\lambda^2[n^2(F_{11} + F_{22} + F_{33}) - F_{11} - 2nF_{23}] \\
H_4 &= n^2(F_{12}^2 + F_{13}^2 + F_{23}^2 - F_{11}F_{22} - F_{22}F_{33} - \\
&\quad F_{11}F_{33}) + 2n(F_{11}F_{23} - F_{12}F_{13}) + 2\lambda(F_{12}F_{23} - \\
&\quad F_{13}F_{22}) \\
H_5 &= 2n\lambda(F_{12}^2 + F_{13}^2 + F_{23}^2 - F_{11}F_{22} - F_{22}F_{33} - \\
&\quad F_{11}F_{33}) + 4\lambda(F_{11}F_{23} - F_{12}F_{13}) \\
H_6 &= F_{11}F_{22}F_{33} + 2F_{12}F_{13}F_{23} - F_{11}F_{23}^2 - F_{22}F_{13}^2 - \\
&\quad F_{33}F_{12}^2 \\
H_7 &= \lambda^4(F_{11} + F_{22} + F_{33}) \\
H_8 &= 2n^2\lambda^2(F_{11} + F_{22} + F_{33}) + 2\lambda^3 F_{13} - 2n\lambda^2 F_{23} \\
H_9 &= 4\lambda^3[n(F_{11} + F_{22} + F_{33}) - F_{23}] \\
H_{10} &= \lambda^2(F_{12}^2 + F_{23}^2 + F_{13}^2 - F_{11}F_{22} - F_{22}F_{33} - F_{11}F_{33})
\end{aligned}
\tag{20}
$$

With given material and dimensions, the minimum q_i may be found from Eq. (19) in terms of q_j and q_k, where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$.

If the given material is isotropic, then the characteristic Eq. (18) can be reduced (see Appendix) to the characteristic Eqs. (VII-10) and (VII-24) given by Flugge.[6] For orthotropic material, the characteristic Eq. (18) does not check exactly the characteristic equation by Cheng.[3] The latter is not symmetric because the basic relations and equations are due to Timoshenko and Gere[10] with more simplifications involved. However, their numerical results have no significant difference, as will be pointed out in another paper.[16]

Boundary Conditions

As was stated in the previous section, if the cylinder is not very long, the constraint effect at the ends cannot be neglected. Furthermore, when a very long cylinder is being discussed, it is difficult to decide what length may be called safely "very long." Hence, it is essential to investigate the boundary conditions.

Take another look at the characteristic Eq. (18). After expanding, it is an eighth-degree polynomial of λ. Assume that the minimum q_i, corresponding to a certain value of n, has been found from Eq. (19) in terms of q_j and q_k, where $i, j, k = 1, 2, 3$ and $i \neq j \neq k$. Substituting q_i, q_j, q_k, and n, as well as the given dimensions (except length) and material properties in the polynomial, eight roots of λ will be found. At least two of these roots are real, because one of them must be the original real one. All eight roots of λ satisfy the characteristic Eq. (18) with the same q_i, q_j, q_k, and n. By the principle of superposition, Eqs. (15) become‡

‡ This method first was used by Donnell, in 1934, to solve the buckling of thin tube under torsion.[5] He simplified Flugge's differential equations to such a point that the expanded characteristic equation was a fourth-degree polynomial of λ. He had to do that because, at this time, without the aid of a high speed electronic computer, it was not realistic to attempt to solve an eighth-degree polynomial.

$$
\begin{aligned}
u &= \sum_{k=1}^{8} U_k \sin\left(\frac{\lambda_k x}{a} + n\theta\right) \\
v &= \sum_{k=1}^{8} V_k \sin\left(\frac{\lambda_k x}{a} + n\theta\right) \\
w &= \sum_{k=1}^{8} W_k \cos\left(\frac{\lambda_k x}{a} + n\theta\right)
\end{aligned}
\tag{21}
$$

For each λ_k, U_k and V_k can be determined from Eqs. (16) in terms of W_k. Let the root λ_k be

$$\lambda_k = (\lambda_r + i\lambda_i)_k \qquad i = -1^{1/2} \tag{22}$$

Then, correspondingly,

$$W_k = (W_r + iW_i)_k \tag{23}$$

$$U_k = (f_{ur} + if_{ui})_k W_k \tag{24}$$

$$V_k = (f_{vr} + if_{vi})_k W_k \tag{25}$$

where λ_r, λ_i, W_r, W_i, f_{ur}, f_{ui}, f_{vr}, and f_{vi} are all real values. The coefficients f_{ur}, f_{ui}, f_{vr}, and f_{vi} are determined from Eqs. (16). If λ_k is real, then λ_i, W_i, f_{ui}, and f_{vi} will be zero. In defining, if $\lambda_i = 0$, then

$$W_{r_k} = \bar{W}_k \qquad W_{r_{k+1}} = \bar{W}_{k+1}$$

and if $\lambda_i \neq 0$, then

$$W_{r_k} = \quad W_{r_{k+1}} = \tfrac{1}{2}\bar{W}_k$$

$$W_{i_k} = -W_{i_{k+1}} = \tfrac{1}{2}\bar{W}_{k+1}$$

After a few manipulations, Eqs. (21) become

$$
\begin{aligned}
u &= \sum_{k=1,3,}^{7} [(U_{cr_k}\bar{W}_k + U_{ci_{k+1}}\bar{W}_{k+1}) \cos n\theta + \\
&\quad (U_{sr_k}\bar{W}_k + U_{si_{k+1}}\bar{W}_{k+1}) \sin n\theta] \\
v &= \sum_{k=1,3,}^{7} [(V_{cr_k}\bar{W}_k + V_{ci_{k+1}}\bar{W}_{k+1}) \cos n\theta + \\
&\quad (V_{sr_k}\bar{W}_k + V_{si_{k+1}}\bar{W}_{k+1}) \sin n\theta] \\
w &= \sum_{k=1,3,}^{7} [(W_{cr_k}\bar{W}_k + W_{ci_{k+1}}\bar{W}_{k+1}) \cos n\theta + \\
&\quad (W_{sr_k}\bar{W}_k + W_{si_{k+1}}\bar{W}_{k+1}) \sin n\theta]
\end{aligned}
\tag{26}
$$

where

$$
\begin{aligned}
U_{cr} &= f_{ur}\zeta_3 - f_{ui}\zeta_4 \\
U_{sr} &= f_{ur}\zeta_2 + f_{ui}\zeta_1 \\
V_{cr} &= f_{vr}\zeta_3 - f_{vi}\zeta_4 \\
V_{sr} &= f_{vr}\zeta_2 + f_{vi}\zeta_1 \\
W_{cr} &= \zeta_2 \\
W_{sr} &= -\zeta_3 \\
\zeta_1 &= \sin(\lambda_r x/a) \sinh(\lambda_i x/a) \\
\zeta_2 &= \cos(\lambda_r x/a) \cosh(\lambda_i x/a) \\
\zeta_3 &= \sin(\lambda_r x/a) \cosh(\lambda_i x/a) \\
\zeta_4 &= \cos(\lambda_r x/a) \sinh(\lambda_i x/a)
\end{aligned}
\tag{27}
$$

and if $\lambda_i = 0$, then

$$
\begin{aligned}
U_{ci} &= U_{cr} & U_{si} &= U_{sr} \\
V_{ci} &= V_{cr} & V_{si} &= V_{sr} \\
W_{ci} &= W_{cr} & W_{si} &= W_{sr}
\end{aligned}
\tag{28}
$$

and if $\lambda_i \neq 0$, then

$$
\begin{aligned}
U_{ci} &= f_{ur}\zeta_4 + f_{ui}\zeta_3 \\
U_{si} &= -f_{ur}\zeta_1 + f_{ui}\zeta_2 \\
V_{ci} &= f_{vr}\zeta_4 + f_{vi}\zeta_3 \\
V_{si} &= -f_{vr}\zeta_1 + f_{vi}\zeta_2 \\
W_{ci} &= -\zeta_1 \\
W_{si} &= -\zeta_4
\end{aligned}
\tag{29}
$$

Substituting Eqs. (26) in Eqs. (7) and (8), one has

$$
\begin{aligned}
w_{,x} &= \frac{1}{a}\sum_{k=1,3,}^{7} [(W_{xcr_k}\bar{W}_k + W_{xci_{k+1}}\bar{W}_{k+1})\cos n\theta + (W_{xsr_k}\bar{W}_k + W_{xsi_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
e_x^0 &= \frac{1}{a}\sum_{k=1,3,}^{7} [(e_{xcr_k}\bar{W}_k + e_{xci_{k+1}}\bar{W}_{k+1})\cos n\theta + (e_{xsr_k}\bar{W}_k + e_{xsi_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
e_\theta^0 &= \frac{1}{a}\sum_{k=1,3,}^{7} [(e_{\theta cr_k}\bar{W}_k + e_{\theta ci_{k+1}}\bar{W}_{k+1})\cos n\theta + (e_{\theta sr_k}\bar{W}_k + e_{\theta si_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
e_{x\theta}^0 &= \frac{1}{a}\sum_{k=1,3,}^{7} [(e_{x\theta cr_k}\bar{W}_k + e_{x\theta ci_{k+1}}\bar{W}_{k+1})\cos n\theta + (e_{x\theta sr_k}\bar{W}_k + e_{x\theta si_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
X_x &= \frac{1}{a^2}\sum_{k=1,3,}^{7} [(X_{xcr_k}\bar{W}_k + X_{xci_{k+1}}\bar{W}_{k+1})\cos n\theta + (X_{xsr_k}W_k + X_{xsi_{k+1}}W_{k+1})\sin n\theta] \\
X_\theta &= \frac{1}{a^2}\sum_{k=1,3,}^{7} [(X_{\theta cr_k}\bar{W}_k + X_{x\theta ci_{k+1}}\bar{W}_{k+1})\cos n\theta + (X_{\theta sr_k}\bar{W}_k + X_{\theta si_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
X_{x\theta} &= \frac{1}{a^2}\sum_{k=1,3,}^{7} [(X_{x\theta cr_k}\bar{W}_k + X_{x\theta ci_{k+1}}\bar{W}_{k+1})\cos n\theta + (X_{x\theta sr_k}\bar{W}_k + X_{x\theta si_{k+1}}\bar{W}_{k+1})\sin n\theta]
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
W_{xcr} &= -\lambda_r\zeta_3 + \lambda_i\zeta_4 \\
W_{xsr} &= -\lambda_r\zeta_2 - \lambda_i\zeta_1 \\
e_{xcr} &= \lambda_r U_{sr} - \lambda_i U_{si} \\
e_{xsr} &= -\lambda_r U_{cr} + \lambda_i U_{ci} \\
e_{\theta cr} &= nV_{sr} + W_{cr} \\
e_{\theta sr} &= -nV_{cr} + W_{sr} \\
e_{x\theta cr} &= nU_{sr} + \lambda_r V_{sr} - \lambda_i V_{si} \\
e_{x\theta sr} &= -nU_{cr} - \lambda_r V_{cr} + \lambda_i V_{ci} \\
X_{xcr} &= (\lambda_r^2 - \lambda_i^2)\zeta_2 + 2\lambda_r\lambda_i\zeta_1 \\
X_{xsr} &= -(\lambda_r^2 - \lambda_i^2)\zeta_3 + 2\lambda_r\lambda_i\zeta_4 \\
X_{\theta cr} &= (n^2 - 1)\zeta_2 \\
X_{\theta sr} &= -(n^2 - 1)\zeta_3 \\
X_{x\theta cr} &= -n(2W_{xsr} + U_{sr}) + \lambda_r V_{sr} - \lambda_i V_{si} \\
X_{x\theta sr} &= n(2W_{xcr} + U_{cr}) - \lambda_r V_{cr} + \lambda_i V_{ci}
\end{aligned}
\tag{31}
$$

and if $\lambda_i = 0$, then

$$
\begin{aligned}
W_{xci} &= W_{xcr} & W_{xsi} &= W_{xsr} \\
e_{xci} &= e_{xcr} & e_{xsi} &= e_{xsr} \\
e_{\theta ci} &= e_{\theta cr} & e_{\theta si} &= e_{\theta sr} \\
e_{x\theta ci} &= e_{x\theta cr} & e_{x\theta si} &= e_{x\theta sr} \\
X_{xci} &= X_{xcr} & X_{xsi} &= X_{xsr} \\
X_{\theta ci} &= X_{\theta cr} & X_{\theta si} &= X_{\theta sr} \\
X_{x\theta ci} &= X_{x\theta cr} & X_{x\theta si} &= X_{x\theta sr}
\end{aligned}
\tag{32}
$$

and if $\lambda_i \neq 0$, then

$$
\begin{aligned}
W_{xci} &= -\lambda_r\zeta_4 - \lambda_i\zeta_3 \\
W_{xsi} &= \lambda_r\zeta_1 - \lambda_1\zeta_2 \\
e_{xci} &= \lambda_r U_{si} + \lambda_i U_{sr} \\
e_{xsi} &= -\lambda_r U_{ci} - \lambda_i U_{cr} \\
e_{\theta ci} &= nV_{si} + W_{ci} \\
e_{\theta si} &= -nV_{ci} + W_{si} \\
e_{x\theta ci} &= nU_{si} + \lambda_r V_{si} + \lambda_i V_{sr} \\
e_{x\theta si} &= -nU_{ci} - \lambda_r V_{ci} - \lambda_i V_{cr} \\
X_{xci} &= -(\lambda_r^2 - \lambda_i^2)\zeta_1 + 2\lambda_r\lambda_i\zeta_2 \\
X_{xsi} &= -(\lambda_r^2 - \lambda_i^2)\zeta_4 - 2\lambda_r\lambda_i\zeta_3 \\
X_{\theta ci} &= -(n^2 - 1)\zeta_1 \\
X_{\theta si} &= -(n^2 - 1)\zeta_4 \\
X_{x\theta ci} &= -n(2W_{xsi} + U_{si}) + \lambda_r V_{si} + \lambda_i V_{sr} \\
X_{x\theta si} &= n(2W_{xci} + U_{ci}) - \lambda_r V_{ci} - \lambda_i V_{cr}
\end{aligned}
\tag{33}
$$

With all the foregoing equations available, the investigation of the boundary conditions now may proceed. It has been pointed out in many textbooks[6, 14] that, at the boundary of a cylindrical shell, the shearing force $N_{x\theta}$ and the twisting moment $M_{x\theta}$ can be combined together to be an effective shear per unit length T_x, i.e.,

$$T_x = N_{x\theta} + (M_{x\theta}/a) \tag{34}$$

If there exists an axial load P due to the rotation of a line element by an angle $v_{,x}$, this axial load P produces a component $Pv_{,x}$ in the circumferential direction. Therefore, the total shear per unit length T_{x_t} at both ends of the cylinder is

$$T_{x_t} = T_x + Pv_{,x} - T \tag{35}$$

By the same reasoning, the torsional load T at both ends produces a component $Tu_{,\theta}/a$ in the axial direction due to the rotation of a line element by an angle $u_{,\theta}/a$.[6] The total axial force per unit length N_{xt} at both ends is

$$N_{xt} = N_x + Tu_{,\theta}/a - P \tag{36}$$

There are four boundary conditions at each end of the cylinder. Two of them are as follows:

1) For the case of both ends hinged

$$w = 0 \qquad M_x = 0 \qquad \text{at } x = \pm(L/2) \tag{37}$$

2) For the case of both ends clamped

$$w = 0 \qquad w_{,x} = 0 \qquad \text{at } x = \pm(L/2) \tag{38}$$

Another pair of boundary conditions for both of the foregoing cases are as follows:

3) If $v, u \neq 0$ at both ends, then

$$
\left.
\begin{aligned}
T_x + Pv_{,x} &= 0 \\
N_x + Tu_{,\theta}/a &= 0
\end{aligned}
\right\} \text{at } x = \pm(L/2) \tag{39}
$$

4) If $v, u = 0$ at both ends, then

$$v = 0 \qquad u = 0 \qquad \text{at } x = \pm(L/2) \tag{40}$$

5) If $v \neq 0$ and $u = 0$ at both ends, then

$$
\left.
\begin{aligned}
T_x + Pv_{,x} &= 0 \\
u &= 0
\end{aligned}
\right\} \text{at } x = \pm(L/2) \tag{41}
$$

6) If $v = 0$ and $u \neq 0$ at both ends, then

$$
\left.
\begin{aligned}
v &= 0 \\
N_x + Tu_{,\theta}/a &= 0
\end{aligned}
\right\} \text{at } x = \pm(L/2) \tag{42}
$$

For example, for radial load only, it is known that $P = T = 0$ and $v, u \neq 0$. The boundary conditions are Eqs. (37) and (39) for both ends hinged, and Eqs. (38) and (39) for both ends clamped.

From Eqs. (9, 30, and 34), one has

$$
\begin{aligned}
N_x &= \frac{A_{22}}{a}\sum_{k=1,3,}^{7} [(N_{xcr_k}\bar{W}_k + N_{xci_{k+1}}\bar{W}_{k+1})\cos n\theta + (N_{xsr_k}\bar{W}_k + N_{xsi_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
T_x &= \frac{A_{22}}{a}\sum_{k=1,3,}^{7} [(T_{xcr_k}\bar{W}_k + T_{xci_{k+1}}\bar{W}_{k+1})\cos n\theta + (T_{xsr_k}\bar{W}_k + T_{xsi_{k+1}}\bar{W}_{k+1})\sin n\theta] \\
M_x &= A_{22}\sum_{k=1,3,}^{7} [(M_{xcr_k}\bar{W}_k + M_{xci_{k+1}}\bar{W}_{k+1})\cos n\theta + (M_{xsr_k}\bar{W}_k + M_{xsi_{k+1}}\bar{W}_{k+1})\sin n\theta]
\end{aligned}
\tag{43}
$$

where

$$\begin{aligned}
N_{xj} &= (\bar{A}_{11} + \bar{B}_{11})e_{xj} + (\bar{A}_{12} + \bar{B}_{12})e_{\theta j} + [\bar{A}_{16} + \bar{B}_{16} + (\bar{D}_{16}/2)]e_{x\theta j} + (\bar{B}_{11} + \bar{D}_{11})X_{xj} + \bar{B}_{12}X_{\theta j} + [\bar{B}_{16} + (\bar{D}_{16}/2)]X_{x\theta j} \\
T_{xj} &= (\bar{A}_{16} + 2\bar{B}_{16} + \bar{D}_{16})e_{xj} + (\bar{A}_{26} + 2\bar{B}_{26} + \bar{D}_{26})e_{\theta j} + (\bar{A}_{66} + 2\bar{B}_{66} + \tfrac{3}{2}\bar{D}_{66})e_{x\theta j} + (\bar{B}_{16} + 2\bar{D}_{16})X_{xj} + (\bar{B}_{26} + \bar{D}_{26})X_{\theta j} + (\bar{B}_{66} + \tfrac{3}{2}\bar{D}_{66})X_{x\theta j} \\
M_{xj} &= (\bar{B}_{11} + \bar{D}_{11})e_{xj} + (\bar{B}_{12} + \bar{D}_{12})e_{\theta j} + (\bar{B}_{16} + \bar{D}_{16})e_{x\theta j} + \bar{D}_{11}X_{xj} + \bar{D}_{12}X_{\theta j} + \bar{D}_{16}X_{x\theta j}
\end{aligned} \quad (44)$$

with $j = cr, ci, sr, si$. From Eqs. (26), (43), and the first of Eqs. (30), it is clear that, corresponding to conditions of Eqs. (37) and (39), at $x = L/2$,§ one has

$$\begin{aligned}
\sum_{k=1,3,}^{7} (W_{cr_k}\bar{W}_k + W_{ci_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (W_{sr_k}\bar{W}_k + W_{si_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (T_{xcr_k}\bar{W}_k + T_{xci_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (T_{xsr_k}\bar{W}_k + T_{xsi_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (N_{xcr_k}\bar{W}_k + N_{xci_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (N_{xsr_k}\bar{W}_k + N_{xsi_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (M_{xcr_k}\bar{W}_k + M_{xci_{k+1}}\bar{W}_{k+1}) &= 0 \\
\sum_{k=1,3,}^{7} (M_{xsr_k}\bar{W}_k + M_{xsi_{k+1}}\bar{W}_{k+1}) &= 0
\end{aligned} \quad (45)$$

Equations for other conditions can be formulated similarly. Equations (45) or their equivalents for different boundary conditions are eight homogeneous equations of eight unknown $\bar{W}$'s. For the nontrivial solution, the determinant of the coefficients (let it be called boundary characteristic) must equal zero. The minimum length L, except in the case of the identity $L = 0$, solved from the zero determinant, is the length of the cylinder corresponding to the given q_1, q_2, q_3, n, h, a, boundary conditions, and material properties. Repeating the foregoing procedure with different original length, curves can be constructed which represent critical loads as functions of the geometrical dimensions and material properties of the cylindrical shell.

The details of methods by which the results obtained in the present paper may be applied as well as some numerical examples and the effect of the boundary conditions on the buckling load will be discussed in another paper.[16]

Appendix: Reduction of the Buckling Characteristic Eq. (18) to the Case for Isotropic Material

The elastic moduli of Eq. (2) for an isotropic material are

$$[C] = [E/(1-\nu^2)][\bar{C}] \quad (A1)$$

§ $x = L/2$ or $x = -L/2$ will not make any difference because the functions are either a purely odd function of x or a purely even function of x.

where

E = Young's modulus

ν = Poisson's ratio

$$[\bar{C}] = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \quad (A2)$$

Then, from Eqs. (10) and (13), one has

$$\begin{aligned}
[A] &= [Eh/(1-\nu^2)][\bar{C}] \\
[B] &= [0] \\
[D] &= [Eh^3/12(1-\nu^2)][\bar{C}]
\end{aligned} \quad (A3)$$

and

$$\begin{aligned}
[\bar{A}] &= [\bar{C}] \\
[\bar{B}] &= [0] \\
[\bar{D}] &= \alpha[\bar{C}]
\end{aligned} \quad (A4)$$

where $\alpha = h^2/12a^2$. Substituting Eqs. (A4) in Eqs. (17), one has

$$(F) = \begin{bmatrix}
\lambda^2 + \frac{1-\nu}{2}(1+\alpha)n^2 & \frac{1+\nu}{2}n\lambda & \alpha\lambda^3 + \left(\nu - \frac{1-\nu}{2}n^2\alpha\right)\lambda \\
\frac{1+\nu}{2}n\lambda & \frac{1-\nu}{2}(1+3\alpha)\lambda^2 + n^2 & \frac{3-\nu}{2}\alpha\lambda^2 + n \\
\alpha\lambda^3 + \left(\nu - \frac{1-\nu}{2}n^2\alpha\right)\lambda & \frac{3-\nu}{2}\alpha\lambda^2 + n & \alpha\lambda^4 + 2n^2\alpha\lambda^2 + (n^2-1)^2\alpha + 1
\end{bmatrix} \quad (A5)$$

Substituting Eq. (A5) in Eq. (18), one has the buckling characteristic equation for isotropic material:

$$\begin{vmatrix}
\lambda^2 + \frac{1-\nu}{2}(1+\alpha)n^2 - n^2q_1 - \lambda^2q_2 - 2n\lambda q_3 & \frac{1+\nu}{2}n\lambda & \alpha\lambda^3 + \left(\nu - \frac{1-\nu}{2}n^2\alpha\right)\lambda + \lambda q_1 \\
\frac{1+\nu}{2}n\lambda & \frac{1-\nu}{2}(1+3\alpha)\lambda^2 + n^2 - n^2q_1 - \lambda^2q_2 - 2n\lambda q_3 & \frac{3-\nu}{2}n\alpha\lambda^2 + n - nq_1 - 2\lambda q_3 \\
\alpha\lambda^3 + \left(\nu - \frac{1-\nu}{2}n^2\alpha\right)\lambda + \lambda q_1 & \frac{3-\nu}{2} \times n\alpha\lambda^2 + n - nq_1 - 2\lambda q_3 & \alpha\lambda^4 + 2n^2\alpha\lambda^2 + (n^2-1)^2\alpha + 1 - n^2q_1 - \lambda^2q_2 - 2n\lambda q_3
\end{vmatrix} = 0 \quad (A6)$$

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APRIL 1963 AIAA JOURNAL VOL. 1, NO. 4

Some Recent Contributions to Panel Flutter Research

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With the objective of formulating a realistic computing program to analyze panel flutter in aerospace vehicles, plausible simplifying assumptions are examined in the light of experimental results. It is shown that in certain areas very simple analysis yields respectable results, whereas in other areas great elaboration is necessary to obtain an accurate prediction. In particular, the role played by the boundary layer flow is discussed. The attenuation and phase shift in pressure-deflection relationship caused by the boundary layer can become important under certain circumstances. Examples are given which show that the boundary layer greatly stabilizes flat plates in a transonic or low supersonic flow and circular cylindrical shells at higher Mach numbers. Some recent contributions to panel flutter research by the author and his colleagues and students at the California Institute of Technology are summarized. Although details are to be published elsewhere, a brief description of experimental results concerning flat plates and cylindrical shells is given here. The experimental and theoretical investigations taken together provide a fairly clear picture with regard to proper assumptions for an accurate analysis. Recommendations for future research in this field are given.

Nomenclature

A = $\rho U^2 L^3/MD$, ratio of dynamic pressure to panel rigidity = π^4 (Q of Ref. 1)
A_ν = coefficients of Fourier series of $z_0(x,t)$, Eq. (9)
a_m = $m = 1, 2, \ldots$, coefficients of sine series of $z_0(x,t)$, Eqs. (29) and (30)
a, a_δ = velocity of sound, in main flow and boundary layer, respectively
B_ν = coefficients of Fourier series of $z_1(x,t)$, Eq. (10)
C_ν, D_ν, E_ν = coefficients, see Eqs. (12) and (13)
D = $Eh^3/[12(1 - \mu^2)]$, bending rigidity of plate
f = frequency, cps
g = structural damping factor
h = thickness of plate or shell wall
k = $\omega L/U$, reduced frequency in main flow
k_δ = $\omega L/U_\delta$, reduced frequency in boundary layer
L = chord length
M, M_δ = Mach number of main flow and of boundary layer, respectively
n = number of waves along circumference (number of nodes = $2n$)
Δp = see Eq. (33)
$p(x,t)$ = wall pressure
p, p_δ = static pressure in freestream and in boundary layer, respectively
p_m = excess of model internal pressure above p_δ, psig
p_t = wind tunnel stagnation pressure
$p_0(x,t)$ = wall pressure in potential flow without boundary layer
q = $\frac{1}{2}\rho U^2$, dynamic pressure of main flow
R = radius of middle surface of circular cylinder
r,θ,x = cylindrical polar coordinates
T,T_δ = absolute temperature in freestream and in boundary layer, respectively

Presented at the IAS 31st Annual Meeting, New York, January 21–23, 1963; revision received March 4, 1963. This research was supported by the U. S. Air Force through the Office of Scientific Research, Office of Aerospace Research. Howard Wolko and Joseph Long of the Office of Scientific Research provided much help to the author. The experimental work at NASA Ames Research Center was carried out with the help and guidance of Lloyd Jones and Ralph Huntsberger; to them and to the staff of the Unitary Plan wind tunnel, the author is sincerely grateful. The Guggenheim Aeronautical Laboratory, California Institute of Technology team whose work is reported here consists of Hans Krumhaar, William J. Anderson, Ronald O. Stearman, Malcolm Lock, Benjamin Cummings, George Watts, and J. Stuart Keith; to their contributions, individual references are made in this paper. Ernest E. Sechler provided much inspiration and guidance to experimentation. Louis Schmidt and Richard Luntz offered tireless help during the wind tunnel tests. Above all, Marvin Jessey designed and made many of the instrumentation components; to his patience, ability, and accuracy the author owes much for any success of his research. Dorothy Eaton did most of the numerical calculations. Helen Burrus and Jeanette Siefke prepared the manuscript. To all these people and to many others not here mentioned, the author wishes to express his sincere appreciation and thanks.

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